A Note on Circular Arc Online Coloring using First Fit

Paraskevas V. Lekeas

6807 N Keeler Ave, Lincolnwood

IL 60712, USA

plekeas@gmail.com

Abstract

In [1] using a column construction technique it is proved that every interval graph can be colored online with First Fit with at most $8w(G)$ colors, where $w(G)$ is the size of the maximum clique of G. Since the column construction can not be adapted to circular arc graphs we give a different proof to establish an upper bound of $9w(G)$ for online coloring a circular arc graph G with the First Fit algorithm.

1 Introduction

A Circular arc graph is the intersection graph of arcs of a circle [2, 3], while Interval graph is a graph that has an intersection model consisting of intervals on the real line [2]. Circular arc graphs are somehow a generalization of Interval graphs [4, 5]. The problem of online coloring an interval graph using the FirstFit (*FF*) algorithm is the following: The intervals of the graph appear one at a time and for each interval *FF* must assign the least available color[1]. A lot of research has been conducted to bound the maximum number of colors that *FF* will use [1, 6, 7]. In his Ph.D. R. Raman [1] proved that for every interval graph G with maximum clique size $w(G)$, *FF* needs at most $8w(G)$ colors to color online G. Now moving our attention to online coloring circular arc graphs, to the best of our knowledge the performance

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[1]Colors are supposed to be positive integers $1, 2, 3, \cdots$.

of FF has not been studied. In [8] a heuristic algorithm for online coloring a circular arc graph is given and in [9] an algorithm that partially uses FF for online coloring a circular arc graph is used. In this note we prove that if G is a circular arc graph with maximum clique size $w(G)$ then FF needs at most $9w(G)$ colors to color online G. Although this result is very close to Raman's theorem for interval graphs, the construction proposed in [1] can not be adapted to circular arc graphs. The reason is that in circular arc graphs a unique natural ordering of the arcs - crucial for the column construction in Raman's theorem - does not exist. This fact motivated us to find a different way to bound the number of colors used by FF to color online a circular arc graph. In the rest of the note in section 2 we describe the problem of online coloring[2] an interval graph proving a crucial Lemma. In section 3 we describe the problem of online coloring a circular arc graph for the case where the circular arc graph has a minimum clique size of 1. In the last section we discuss the general case where the circular arc graph G contains a minimum clique of size $K \leq w(G)$.

2 Interval graph online coloring using FF

Assume G' is an interval graph with $m+1$ intervals and $w(G')$ the maximum clique size of G'. Let $\Sigma = < \delta_1, \delta_2, ..., \delta_{m+1} >$ be a sequence (ordered $(m+1)$-tuple) denoting the order of appearance of the intervals of G' in a valid online coloring with FF. Assume that the FF algorithm, for every sequence Σ, builds another sequence of positive integers, $X = < \chi_1, \chi_2, ..., \chi_{m+1} >$, which denotes what color was assigned to every corresponding interval of Σ. Let us call X the chromatic sequence of Σ. From Raman's theorem [1] and by definition of the FF algorithm, it holds that, $1 \leq \chi_i \leq 8w(G'), \forall \chi_i \in X, \ 1 \leq i \leq m+1$.

Let G' be an interval graph as before and suppose δ_L is the leftmost interval of G'. Let δ_R also be the rightmost interval of G'. Assume for now that δ_L and δ_R are unique and also assume that these two intervals do not intersect in G' (we will later see what this assumption means for our problem). In Figure 1 we can see G'.

In an online coloring of G', from all the possible, $(m + 1)!$ in number, sequences Σ, we choose all these that have the following characteristic: The intervals δ_L and δ_R appear consecutively, i.e. one after the other.

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[2]From now on online coloring refers to online coloring with FF.
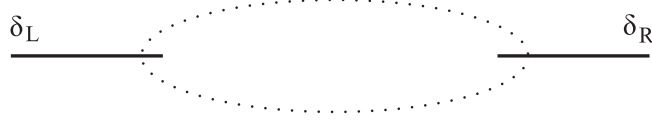
Figure 1: The interval graph G'. Inside the eclipse there exist $m-1$ intervals. Intervals δ_L and δ_R do not overlap.

Let $\Sigma_1 \; =< \delta_1, \delta_2, \cdots, \delta_i, \delta_L, \delta_R, \delta_{i+3}, \cdots, \delta_{m+1} >$ be such a sequence and let $X_1 =< \chi_1, \chi_2, \cdots, \chi_i, \chi_{\delta_L}, \chi_{\delta_R}, \chi_{i+3}, \cdots, \chi_{m+1} >$ be the chromatic sequence that FF produces from Σ_1. As we mentioned earlier, no element of X_1 is bigger than $8w(G')$.

   Assume now that during the online process of coloring Σ_1, FF assigns $\chi_{\delta_L} = L$ and $\chi_{\delta_R} = R$, where L, R are colors $\leq 8\omega(G')$ (it is obvious that there exists such a possible evolution of the online process). We wish to find out how the chromatic sequence X_1 would be affected if someone pauses the online coloring process immediately after the i-th step is completed, assigns $\chi_{\delta_L} = \Delta$ and $\chi_{\delta_R} = \Delta$, where $\Delta = 8w(G') + 1$, and lets it roll again from step $i + 3$. Of course the part of X_1 that would get affected would be the subsequence $< \chi_{i+3}, \chi_{i+4}, ..., \chi_{m+1} >$ and this is because for all the previous intervals a valid color has been assigned and these assignments are permanent according to the definition of FF. Let $< \chi'_{i+3}, \chi'_{i+4}, ..., \chi'_{m+1} >$ be the chromatic subsequence that FF would produce from $(i+3)$-th step (including) and onwards if we didn't pause the process after the end of the i-th step (let us call this process the non-pause process). We have now the following

   Lemma 1: $\chi_j \leq \chi'_j \leq 8w(G')$, $\forall j$, $i + 3 \leq j \leq m + 1$.
   Proof: Assume that $X'_1 =< \chi'_1, \cdots, \chi'_i, \chi'_{\delta_L}, \chi'_{\delta_R}, \chi'_{i+3}, \cdots, \chi'_j, \cdots, \chi'_{m+1} >$ is the chromatic sequence in the non-pause process and
   $X_1 \;=< \chi_1, \cdots, \chi_i, \chi_{\delta_L}, \chi_{\delta_R}, \chi_{i+3}, \cdots, \chi_j, \cdots, \chi_{m+1} >$ is the chromatic sequence in the pause process. The proof is by induction on j.
   <u>Base</u>: $j = i + 3$. Assume that in the non-pause process we had that $\chi'_{\delta_L} = L$, $\chi'_{\delta_R} = R$ and $\chi'_{i+3} = \Pi$. We have the following cases:

- δ_{i+3} does not overlap with either δ_L or δ_R. Since δ_{i+3} received in the non-pause process color Π, this means that Π was the first (smaller) available color. In the pause process now the overlapping constraints for δ_{i+3} are exactly the same with the non-pause process and since

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$\chi_1 = \chi'_1, \cdots, \chi_i = \chi'_i$, Π is still the smaller color available for δ_{i+3}, so $\chi_{i+3} = \chi'_{i+3}$ and thus $\chi_{i+3} \leq \chi'_{i+3}$ as desired.

- δ_{i+3} overlaps either with δ_L, or with δ_R, or with both, then in any case we have the following.

    - if in the non-pause process it is the case that $\Pi < R$ and $\Pi < L$, then after the pause we will have that $\chi_{i+3} = \chi'_{i+3} = \Pi$. This is because by assigning in the pause process the color $\Delta = 8\omega(G')+1$ to δ_L and δ_R we have not ruled out the availability of color Π for interval δ_{i+3}. And since Δ is greater than both L and R, $\chi_1 = \chi'_1, \cdots, \chi_i = \chi'_i$ and the overlapping constraints for δ_{i+3} are the same in both processes, then Π is the least color available for δ_{i+3}. So we have again that $\chi_{i+3} \leq \chi'_{i+3}$.

    - If in the non-pause process it is the case that $R < \Pi$ and $L < \Pi$, then we know for sure that Π is available because $\chi'_{i+3} = \Pi$. We also know that since colors L and R are not assigned anymore to δ_L and δ_R they may be available for δ_{i+3} depending on its neighbors. In any case we will have that χ_{i+3} is going to be the least available color from the set $\{L, R, \Pi\}$. So again $\chi_{i+3} \leq \chi'_{i+3}$.

    - If $R < \Pi < L$, then as before Π is available for δ_{i+3} and L, R may or may not be available depending on the constraints. In any case χ_{i+3} is going to be the least available color from the set $\{R, \Pi\}$. So again $\chi_{i+3} \leq \chi'_{i+3}$.

<u>Induction Hypothesis</u>: $\chi_l \leq \chi'_l, \ \forall l, \ i+3 \leq l \leq j$

<u>Induction Step</u>: We want to prove that $\chi_{j+1} \leq \chi'_{j+1}$. Assume that in the non-pause process $\chi'_{j+1} = \Pi$. Suppose for the sake of contradiction that $\chi_{j+1} > \Pi$. This means that in the pause process there exists a neighbor δ_h of δ_{j+1} that appeared after δ_i with $h < j+1$ and was assigned the color $\chi_{\delta_h} = \Pi$. Now since $h \leq j$ and $\chi'_{\delta_h} \neq \Pi$, from induction hypothesis we have that χ_{δ_h} is strictly less than χ'_{δ_h} ($\chi_{\delta_h} < \chi'_{\delta_h}$). But this leads to a contradiction because of the following reason: Let $\chi'_{\delta_h} > \Pi$. This means that in the non-pause process Π was not available for δ_h, which means that δ_h had a neigbor δ_g with $g < h$ such that $\chi'_{\delta_g} = \Pi$. But from induction hypothesis this means that $\chi_{\delta_g} < \Pi$. So we discovered another interval δ_g for which χ_{δ_g} is strictly less than χ'_{δ_g} ($\chi_{\delta_g} < \chi'_{\delta_g}$). Someone can continue this process and either

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discover an infinite number of intervals appearing after i that satisfy this strictness in the inequalities (which is a contradiction because the process is finite), or discover an interval δ_f for which $f < i$ and $\chi_{\delta_f} < \chi'_{\delta_f}$ which is also a contradiction since we know that $\chi_1 = \chi'_1, \cdots, \chi_i = \chi'_i$. So the assumption that $\chi_{j+1} > \Pi$ is false and we have that $\chi_{j+1} \leq \chi'_{j+1}$.

3   Circular arc graph online coloring using FF

Let us now turn our attention to circular arc graph online coloring using FF. We have the following

Proposition 1: A Circular arc graph G with minimum clique size 1 and maximum clique size $w(G)$ can be colored online by FF with at most $8w(G) + 1$ colors.

Proof: Suppose G is a circular arc graph containing an arc δ such that if removed G can be converted to an interval graph[3]. Assume that $w(G)$ is the maximum clique size of G and there exist m arcs in G, none of which extends $360°$ or more. We can see one such G in Figure 2.
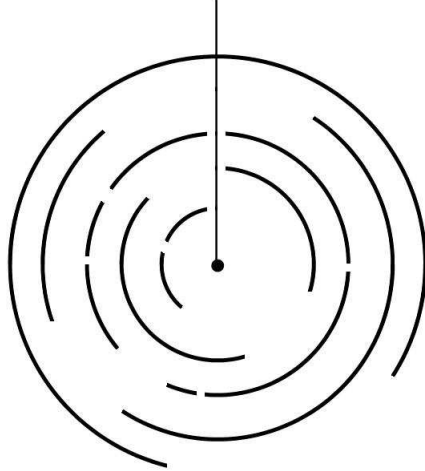

Figure 2: A circular arc graph G with minimum clique of size 1.

Suppose G'' is the interval graph that results if we "cut", G in δ and loosely speaking "unfold" it in the cut. It is obvious that for G'' it holds

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[3]A different definition of G would be that G has a minimum clique of size 1.

$w(G'') = w(G)$. Also G'' has $m + 1$ intervals because when arc δ was "cut" it split into two intervals, say, δ_L and δ_R. It is obvious that G'' is identical with the interval graph G' of the introduction, so in the rest of the note we will refer to G' as the graph resulting from cutting and unfolding the circular arc graph G. We also have to mention that the restriction we posed in the introduction that intervals δ_L and δ_R should not overlap is equivalent with the fact that in G arc δ does not extend $360°$ or more.

We want to prove that the maximum number of colors that FF will use to color online G is at most $8w(G) + 1$ colors. For this and for our analysis to be easier we think of the process of online coloring the circular arc graph G as the following equivalent[4] process of online coloring graph G': While coloring G, when FF is about to assign a color to arc δ, it goes through the following discrete steps: 1st step is to locate the left neighbors of δ and find the color that can be assigned to δ if δ was overlapping only with these left neighbours - assume that this color is L. 2nd step is to do the same for the right neighbors and assume that it finds R. But since arc δ should get a unique color, finally, the 3rd step would be for FF to assign the least available color χ_δ, greater or equal than both L and R, i.e., $\chi_\delta \geq max\{L, R\}$.

Now while coloring G', these steps are more clear. Since in G' arc δ breaks into two intervals δ_L and δ_R, we can say that there exists an online coloring of G' in which δ_L and δ_R appear sequentially one after the other (say δ_L comes first and δ_R follows), δ_L has to get color L (1st step) and δ_R has to get color R (2nd step). Suppose now that someone pauses the process and finds the appropriate smaller available color χ_δ that can be assigned both in δ_L and in δ_R such as $\chi_\delta \geq max\{L, R\}$, and then allows the process to continue. In this way he have mimicked the process of online coloring circular arc graph G with the process of online coloring interval graph G'.

So using the above analysis every online coloring by FF of an arbitrary sequence of the arcs of G, $\Sigma = < \delta_1, \delta_2, ..., \delta_{j-1}, \delta, \delta_{j+1}, ..., \delta_m >$, can be achieved as follows: While FF online colors sequence Σ, when arc δ appears, FF, before assigning a color, breaks[5] δ into two intervals δ_L and δ_R. It assigns to δ_L a color (say L) and to δ_R a color (say R). But since arc δ must receive a uniqe color, finally, FF must assign at most the color $8w(G) + 1$ to interval

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[4]Two online colorings are equivalent when for the same sequences of intervals they produce the same chromatic sequences.

[5]It is as if in the background FF works with the sequence $\Sigma' = < \delta_1, \delta_2, ..., \delta_{j-1}, \delta_L, \delta_R, \delta_{j+1}, ..., \delta_m >$ in order to decide the unique color arc δ should receive.

δ. This is because there exists a case that the least available color for δ can't be any from the set $\{1, \cdots, 8w(G)\}$. Consider for example the case where $\chi_{\delta_L} = 8w(G) - 1$, $\chi_{\delta_R} = 8w(G)$ and δ overlaps with an arc that has assigned previously the color $8w(G)$ (see the example below and Figure 3 for such a construction). Clearly in this case the minimum color that satisfies both constraints of χ_{δ_L} and χ_{δ_R} is $8w(G) + 1$. Using now Lemma 1, FF is sure that no other arriving arc after δ will get a color bigger than $8w(G)$, and thus bigger than $8w(G) + 1$, which completes the proof of Proposition 1.

Before moving to the general case let us give the above-mentioned construction that forces FF to assign the extra color $8w(G) + 1$. In Figure 3 we see a circular arc graph G with minimum clique size 1 and maximum clique size $w(G)$ (for example, the arcs $\{c_1, a_2, \cdots, a_{w(G)}\}$ is such a maximum clique). Consider now the following order of appearance of the arcs in an online coloring of G:

$\Sigma \;=< \; b_1, c_1, a_2, a_3, \cdots, a_{w(G)}, c_2, c_3, \cdots, c_{w(G)-1}, b_2, b_3, \cdots, b_{w(G)-2}, \delta \;>.$
The chromatic sequence produced is $X =< 1, 1, 2, 3, \cdots, w(G), 2, 3, \cdots, w(G) - 1, 2, 3, \cdots, w(G) - 2, w(G) + 1 >$ and this is because when FF is ready to assign a color to arc δ it first brakes δ into δ_L, δ_R and assignes $\chi_{\delta_L} = w(G) - 1$ and $\chi_{\delta_R} = w(G)$, but when it trys to merge δ_L and δ_R to form δ no color from the set $\{1, \cdots, w(G)\}$ is available due to overlaping constraints. So the only solution is for color $8w(G) + 1$ to be introduced and assigned to δ.

4 The general case

In the general case graph G would have a maximum clique of size $w(G)$ and a minimum clique of size $K \leq w(G)$, which if we cut we can convert G into an equivalent interval graph G'. We will prove that in this case FF needs at most $8w(G) + K$ colors to color online G. The proof would be by induction on $j \leq K$.

Theorem: Circular arc graph G can be colored online using FF with at most $8w(G) + K$ colors.
Proof: Induction on $j \leq K$.
<u>Base</u>: $j = 1$, then G has a minimum clique of size 1 and it holds from Proposition 1.
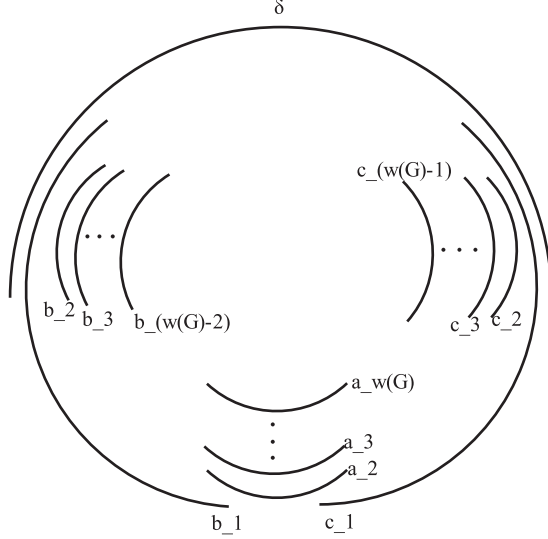
Figure 3: A construction of a circular arc graph G that forces FF to use $8w(G) + 1$ colors in an online coloring.

Induction Hypothesis: Suppose that it holds for j.

Induction Step: Assume G is a circular arc graph with minimum clique of size $j + 1$. Assume that in an online coloring of G the first j arcs of this maximal clique have appeared, and δ is the last arc that needs to appear. Until now we have from the induction hypothesis that no more that $8w(G)+j$ colors have been used. We have to show that from now on at most $8w(G) + j + 1$ colors will be used.

As before we can assume that arc δ can break into two intervals δ_L and δ_R. So the resulting graph is a circular arc graph with minimal clique of size j for which graph the induction hypothesis holds. The process of coloring arc δ can be replaced by the following 3 steps: 1) FF finds the available color from the left constraints, say it is L, 2) FF finds the available color from the right constraints, say it is R and 3) FF finds the minimum available color greater than or equal to L or R and colors δ.

For this case it is also possible to prove a lemma similar to lemma 1 and a proposition similar to proposition 1 and finally applying these prove that the hypothesis also holds for $j + 1$.

Corollary: If G is a circular arc graph with maximal clique $w(G)$, then it can be online colored by FF with at most $9w(G)$ colors.

Proof: We apply the theorem for $K = w(G)$.

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